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The influence of noise on a classical chaotic scatterer

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Abstract

A classical, chaotic scatterer consisting of three, equal-sized, equidistant hard discs, also known as the Gaspard–Rice system [Gaspard P, Rice SA. Scattering from a classically chaotic repellor. J Chem Phys 1989;90(4):2225–2241] is studied in the presence of white noise. The fractal dimension of the stable manifold is measured using the uncertainty fraction. The volume of the manifold, and thus of the invariant set, when considered in a possibilistic sense, is found to scale with the magnitude of the noise, thus extending the results of Ott et al. [Ott E, York ED, Yorke JA. A scaling law: How an attractors volume depends on noise level. Physica D 1985;16:62–78] from attracting to non-attracting sets.

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1. Introduction

The study of nonlinear systems demonstrates a very real limit in our ability to predict the behaviour of real physical systems. With finite knowledge of the system variables, there will

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always be a limit to our predictive ability due to the exponential growth of errors. Even with perfect knowledge (infinite precision) any real system will eventually be perturbed by external factors not accounted for in the model. One of the most common types of perturbation is thermal or Brownian noise which is present to some degree in every physical system. A nonlinear model which includes a noise component would seem to best represent a real world system, such as a turbulent flow, and these types of systems are still not well understood.

The addition of noise to a deterministic system can cause a number of qualitative and quantitative changes in its behaviour. For instance, it can cause a system to escape from its basin of attraction [3], it can create a sensitive dependence where none existed before [4] or a change in the cascade of bifurcations leading to the onset of chaos [5,6]. Of particular interest are the changes in the invariant set of the system [2,7,8]. In [2], Ott et al. demonstrate a scaling law for the volume of an attractor with the magnitude of the noise. In this paper, these results are extended to include a specific class of nonlinear systems containing non-attracting invariant sets—namely, chaotic scatterers.

The system studied consists of three equally sized, equidistant hard discs and is shown in Fig. 1. A point particle is sent into the system with impact parameter b and undergoes perfect elastic collisions with the walls of the discs until it moves past a certain distance from the centre of the system when exit conditions can be recorded. For the numerical simulation, $d = 2.5$ and $r = 1$. All trajectories except a set of measure zero (the chaotic invariant set) will eventually exit the system [9,1].

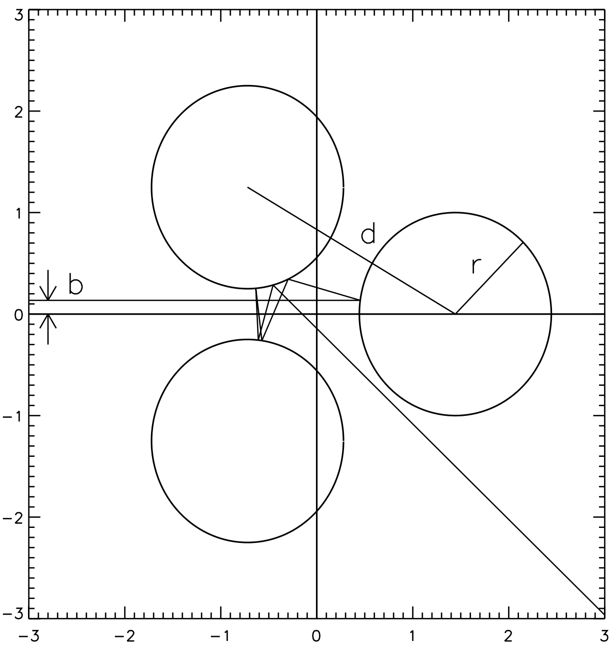

Fig. 1. Three disc scattering system.

2. Fractal dimension

Suppose we describe the system dynamics as an iterated mapping in the phase space, something which in principle at least would be easy to do for this system. Then the invariant set of the system is defined as that set of points which maps back onto itself when the system dynamics are applied. The stable manifold is that set of points which eventually maps onto the invariant set, while the unstable manifold maps onto the invariant set when the system is run backwards in time. From this description it is possible to see that the invariant set is an intersection of the stable and unstable manifolds. In most chaotic systems, the invariant set is fractal. A large portion of chaotic dynamics is concerned with the measurement of the fractal dimension of the chaotic invariant set.

This classical Hamiltonian system is a good example of a hyperbolic chaotic scattering system. In a hyperbolic system, any point in the invariant set can be uniquely represented by a superposition of vectors along the stable and unstable manifolds. Another way of stating this is to say that the intersection between the stable and unstable manifolds which produces the invariant set is generic. A generic intersection between two sets is one which is not qualitatively altered by a small, smooth change in position of either two sets. One of the properties of a hyberbolic scattering system is an exponential decay rate: if a large number of particles are sent into the system, the number of particles remaining in the system will decay exponentially with time [10,9]. This illustrates the most important property of a hyberbolic scatterer: it is completely non-attracting, as there are no tangencies between the stable and unstable manifolds which might form closed orbits.

The system can be converted to an iterated mapping by considering the state of the particle only during a rebound. If this is done, it can be completely represented by only two variables: one angle specifying the position of the particle around the disc during a rebound, and one angle specifying the direction of its travel—due to symmetry, it does not matter which disc the particle is rebounding off. Since the system is Hamiltonian, it is the same whether it is run forwards or backwards: the unstable manifold is simply a mirror image of the stable manifold. Because the intersection of the two sets is generic, the dimension of the chaotic invariant set is given as follows:

$$D = D_s + D_u - N = 2D_s - N = 2(D_s - 1) \tag{1}$$

where D is the fractal dimension of the chaotic invariant set, D_s and D_u are the fractal dimensions of the stable and unstable manifolds respectively and $N = 2$ is the dimensionality of the system. We would expect the stable and unstable manifolds to have a fractal dimension of between one and two. In fact the stable manifold (and therefore the unstable manifold as well) forms an uncountable set of smooth lines whose cross-section forms a cantor set. For a more detailed discussion, see [10].

The fractal dimension of the stable manifold can be calculated by a simple numerical technique known as the uncertainty exponent. Suppose we send a particle into the system with a given set of initial conditions. For a minute displacement, ϵ, in initial conditions chosen in an arbitrary direction, the orbit will either experience a slight change in final conditions or a large change: that is it will be ϵ-certain or ϵ-uncertain. An easy way to determine ϵ-certainty is to count the number of deflections off the discs—if the numbers are different, then the trajectory is uncertain. Given a set of N randomly chosen initial conditions, a certain fraction, f, of them will be uncertain. We would expect the uncertainty fraction to scale as: $f(\epsilon) \sim \epsilon^{\alpha}$, where α is the uncertainty exponent. The uncertainty exponent is defined as follows:

$$\alpha = \lim_{\epsilon \to 0} \frac{\ln f}{\ln \epsilon} \tag{2}$$

The uncertainty dimension is given by $D_s = N - \alpha$ where N is the dimensionality of the system [9–11]. It can be shown that the uncertainty dimension is equivalent to the box-counting or capacity dimension. Let A be a set with capacity dimension D contained in an N dimensional space; then let $B(A, \epsilon)$ be defined as the set formed by expanding each point in A by an amount ϵ. It can be shown that the volume of this set obeys the same scaling law as the uncertainty fraction. Therefore:

$$\mathrm{Vol}[B(A, \epsilon)] \sim f(\epsilon) \approx (k_1 \epsilon)^{N-D} \tag{3}$$

For a more complete discussion, see [2,10].

The fractal dimension may be calculated by fitting Eq. (3) using results from numerical simulations, giving an uncertainty exponent of $\alpha \approx 0.41$. Thus the fractal dimension of the stable manifold is $D_s \approx 1.59$ while the fractal dimension of the chaotic invariant set is $D \approx 1.20$.

3. Adding noise

After each rebound off a disc, the angle of the trajectory was perturbed a random amount. The probability distribution of the random variable was bounded and flat (a.k.a. white noise). When the uncertainty fraction of a system with noise is graphed in Fig. 2, an interesting thing happens. The slope now takes on more than one value: we can pick out a rough transition point in the graph, call it η. At $\epsilon \gtrsim \eta$ the uncertainty exponent is 0, while at $\epsilon \lesssim \eta$ the uncertainty exponent is roughly the same as for the system without noise. We would expect η to be proportional to the standard deviation, σ, of the random variable which is a measure of the magnitude or energy of the noise.

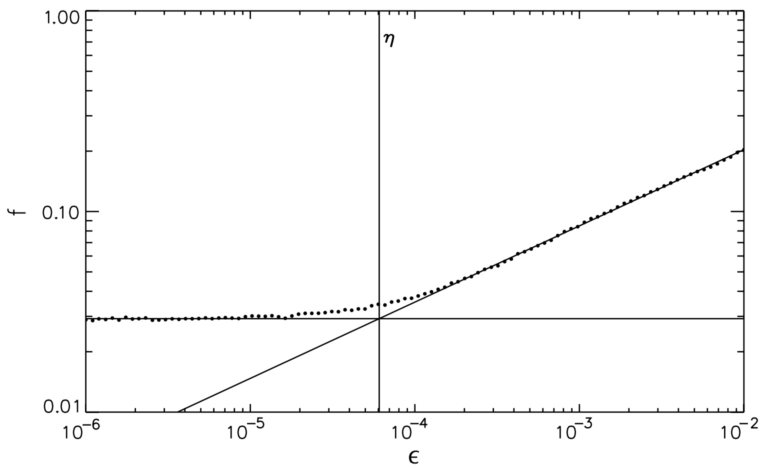

Fig. 2. Uncertainty fraction of a noisy system where $\sigma = 0.001$.

This type of behaviour is also a characteristic of chaotic attractors:

> In the measurement of fractal dimension in experiments it is often important to consider the effect of noise. If we assume the noise is white noise (i.e. it has a flat power spectrum), then we can regard it as essentially fattening (or 'fuzzing') the attractor by an amount of order η, where η represents the noise amplitude. Thus observations of the attractor characteristics on scales $\epsilon > \eta$ the attractor appears to be an N-dimensional volume, where N is the dimension of the space in which the attractor lies [10] (see also for instance, [12]).

Ott et al. have shown that the volume of an attractor with white noise scales as σ^{N-D}, where σ is the magnitude of the noise, as measured by the standard deviation. They reason that after each iteration of the system, the attractor will expand by an amount σ, after which any point outside the attractor must by necessity fall back towards it. Since this is simply an expanded fractal set, Eq. (3) applies [2].

With a scattering system such as ours, we find a similar thing happening. Only in this case we are concerned with the trajectories which can fall into the stable manifold as opposed to those that do. The initial conditions which can fall into the stable manifold would be given by a set $B(A, g(\sigma))$ where A is the stable manifold and g is a function of the magnitude of the noise. From both numerical and theoretical considerations, this should be a straight proportionality: $g(\sigma) \propto \sigma$.

Consider: any trajectory which is a minute distance δ away from the stable manifold should decay away from it at a rate $\sim\delta\lambda$ where λ is the positive Lyapunov exponent of the system. Meanwhile, the noise may be transporting trajectories *towards* the stable manifold at a rate of at most $\sqrt{3}\sigma$. Thus,

$$\frac{\mathrm{d}\delta}{\mathrm{d}n} \approx \lambda\delta - \sqrt{3}\sigma \tag{4}$$

where n is the time dimension and $\delta \geqslant 0$. This defines the approximate transport of trajectories away from the stable manifold at the slowest possible rate. The fixed point is given by:

$$\delta = \frac{\sqrt{3}\sigma}{\lambda} \tag{5}$$

Therefore, once the trajectory has decayed an amount greater than this, it will never enter the stable manifold, provided that the interval between consecutive members is greater than twice this value. The exponential transport of trajectories away from the stable manifold is, in general, much faster than the ability of the noise to shift trajectories toward it.

We can measure the volume of the stable manifold which has been expanded by the noise by the uncertainty fraction at scales less than η. We expect to find the same scaling law as in (3):

$$\mathrm{Vol}[B(A, \sqrt{3}\sigma/\lambda)] \sim f(\epsilon \ll \eta) \approx (k_2\sigma)^{N-D} \tag{6}$$

and fit this curve in Fig. 3 for a 3-disc system with white noise.

It should be noted that the dynamics of the system under investigation are actually a lot simpler than they appear at first glance. If a trajectory does not cross or enter the stable manifold after the first iteration of the system, it is unlikely that it ever will. Note also that unlike in an attracting system, no actual expansion of the chaotic invariant set takes place. Rather it is the probability of a trajectory crossing or entering the set that we are concerned with here. Since the uncertainty fraction is a probabilistic measure, it is naturally affected by this. So the strange behaviour we

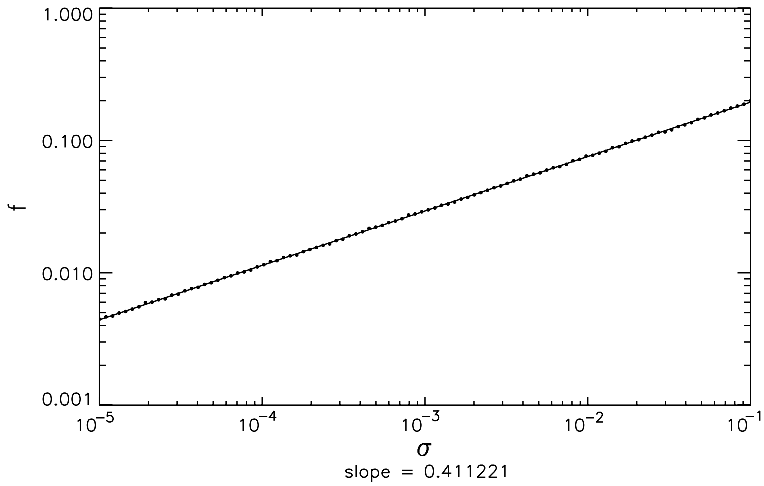

Fig. 3. Noise power versus uncertainty fraction at scales much less than η.

observe with respect to the uncertainty fraction is more the result of the measure we have chosen than a property of the system.

Finally, the uncertainty fraction of the noisy system should be given approximately by:

$$f \approx (k_1\epsilon + k_2\sigma)^{\alpha} \tag{7}$$

Values for the two constants, found by fitting the two curves in (3) and (6) as in Figs. 2 and 3, were $k_1 \approx 2.37$ and $k_2 \approx 0.185$. Fig. 4 is a plot of the uncertainty fraction of a noisy system with a curve obeying the relation overlaid. The transition point, η, may be defined by setting the two terms equal, thus:

$$\eta = \frac{k_1}{k_2}\sigma \tag{8}$$

which can easily be shown to be equivalent to the intersection of two fitted curves as in Fig. 2.

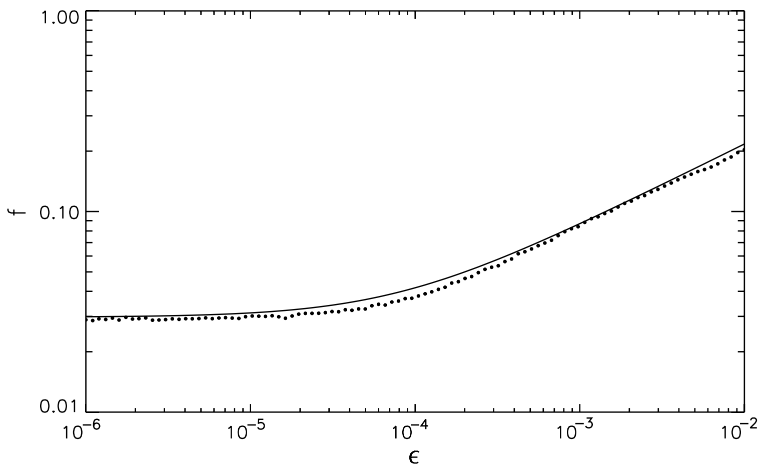

Fig. 4. Uncertainty fraction for noisy system ($\sigma = 0.001$) with curve in (7) overlaid.

4. Discussion

The work done on this system illustrates a simple and (in retrospect) obvious property. For low-dimensional chaotic systems with weak noise, the chaotic behaviour should always dominate over the noisy behaviour above a certain scale. This is because the exponential growth of uncertainties caused by the chaotic dynamics will overwhelm the linear growth of uncertainties caused by the noise.

This work also serves to highlight a limitation of conventional methods when applied to chaotic systems in general and noisy chaotic systems in particular. It is clear that our method of modelling a noisy system is inadequate when we want to rigorously calculate the properties of the system instead of using crude numerical methods. The first thing to understand is that a given simulation represents one possible outcome from many others rather than an exact simulation. The same can be said of a noiseless system, since the numbers used in the simulation are of limited precision. This is another reason why the addition of noise is of relevance: using a point to represent the state of the system is clearly an abstraction. Rather, it might be more realistic to represent the state of the system using a probability distribution and follow how the distribution evolves in time. Such a method is used in [7,8] to directly model noisy chaotic systems.

Lyapunov exponents are an accurate way to characterize the spread of probability distributions in deterministic systems. Unfortunately they are not rigorously defined for noisy systems [5,13] and they are difficult to calculate for systems with many variables. What is needed is some sort of measure to help characterize noisy systems. Good candidates for this are the various measures of entropy for dynamical systems. The metric entropy for example is a single measure which quantifies the amount of information required to track the state of the system over time. While the definition is complicated and will not be stated here (see [10]) it is based on Shannon's measure of information entropy. Unfortunately, the metric entropy is difficult to apply to all but the simplest of systems. Crutchfield and Packard [13] discuss the effect of noise on the metric entropy and find that it scales with the noise magnitude.

A refreshing approach to noisy chaotic systems is given by Shaw [14]. Shaw compares a dynamical system to an information system as described by Shannon and others. This approach takes into account most of the things discussed above. Shaw's methods of depicting dynamical systems have a delightful simplicity and generality which makes them very appealing. They also look easy to convert to numerical methods, at least for simple systems. However, their real world applications may be somewhat limited: rigorous mathematical treatment for instance would be very cumbersome using these methods. In his book, for example, Shaw applies them to a system (a dripping faucet) which is the equivalent of only a one-dimensional function map similar to a quadratic function, such as the logistics equation.

5. Conclusions

White noise was added to a chaotic scattering system composed of three equal-sized, equidistant hard discs. For such a system, the stable manifold as well as the chaotic invariant set may be considered in a possibilistic sense—that is, by focusing on those trajectories that *can* enter it, as opposed to those that actually do. When this is done, the added noise is found to expand the sets, as measured by the uncertainty fraction, by an amount proportional to the magnitude of the noise. Of course the actual probability of any given trajectory entering either the stable manifold or the invariant set is still zero, just as in the non-noisy system.

It is hoped that the focus of new work be on expanding some of the ideas discussed in Section 4, for instance by studying more complicated noisy systems and developing formal and general mathematical treatments for them. The ideal end would be to understand, at an abstract and precise level, highly complex systems in which noise and chaos interact to produce organisation, adaptation and complexity, with living systems being at the furthest end of the scale.

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